Self-duality of the asymptotic relaxation states of fluids and plasmas

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The states of asymptotic relaxation of two-dimensional fluids and plasmas present a high degree of regularity and obedience to the sinh-Poisson equation. We find that by embedding the classical fluid description into a field-theoretical framework, the same equation appears as a manifestation of the self-duality.

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The states generated by externally driving (stirring) an ideal fluid can have a very irregular form. It is, however, known from experiments and numerical simulations that after suppressing the drive the system evolves to states with a high degree of order, essentially consisting of few large vortices, with very regular geometry. These states are attained after long time evolution and are not due to the residual dissipation. The process consists of vortex merging, which is an essentially topological event where the weak dissipation only allows the reconnection of the field lines but does not produce significant energy loss from the fluid motion. Inferring from results of numerical simulations, Montgomery *et al.* [1,2] have proved that the scalar stream function ψ describing the motion in two-dimensional space obeys in the far asymptotic regime (where the regular structures are dominant) the sinh-Poisson equation

$$\Delta\psi+\gamma\sinh(\beta\psi)=0, \tag{1}$$

where γ and β are *positive* constants. The relations of ψ to the velocity and vorticity are $\mathbf{v}=\nabla\psi\times\hat{\mathbf{e}}_z$, $\boldsymbol{\omega}=\nabla\times\mathbf{v}=-\nabla^2\psi\hat{\mathbf{e}}_z$, where $\hat{\mathbf{e}}_z$ is the unitary vector perpendicular to the plane. With these variables, the Euler equations for the two-dimensional ideal incompressible fluids are

$$\nabla\cdot\mathbf{v}=0,\quad \frac{\partial\boldsymbol{\omega}}{\partial t}+(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}=\mathbf{0}. \tag{2}$$

The fact that the sinh-Poisson equation appears in this context is rather unexpected. It is difficult to understand how this equation (exactly integrable and with a very wide involvement in topological soliton and instanton physics) can also describe fluid states. Our objective is to prove that there is a natural way to enlarge the classical Euler fluid description to a field-theoretical framework where the topological properties become more transparent. The most important result is that the asymptotic relaxation states and the sinh-Poisson equation naturally emerge as a consequence of *self-duality*. The extension of the Euler fluid description will be done progressively, examining models elaborated for closely related problems and collecting the relevant suggestions that could allow us to write a Lagrangian density.

In the study of the two-dimensional Euler fluids, and in particular in explaining the origin of Eq. (1), an important model consists of a system of N discrete vorticity filaments perpendicular on plane, having circular transversal section of radius a and carrying the vorticity ω_i, $i=1,N$. A very comprehensive account of this system is given by Kraichnan and Montgomery [3] where the correspondence between the continuous and discrete representations of vorticity is discussed in detail. The motion in plane of the kth filament of coordinates $\mathbf{r}_k\equiv(r_k^1,r_k^2)\equiv(x_k,y_k)$ is given by

$$\frac{dr_k^i}{dt}=\varepsilon^{ij}\frac{\partial}{\partial r_k^j}\sum_{n=1,n\neq k}^{N}\omega_n G(\mathbf{r}_k-\mathbf{r}_n),\quad i,j=1,2,\ k=1,N, \tag{3}$$

where the summation is over all the other filaments' positions $\mathbf{r}_n$, $n\neq k$, and ε^{ij} is the antisymmetric tensor in two dimensions. As shown in Ref. [3] $G(\mathbf{r}_k-\mathbf{r}_n)$ can be approximated for a small compared to the space extension of the fluid, L, $a\ll L$, as the Green function of the Laplacian

$$G(\mathbf{r},\mathbf{r}')\approx-\frac{1}{2\pi}\ln\left(\frac{|\mathbf{r}-\mathbf{r}'|}{L}\right). \tag{4}$$

Using the Liouville theorem and the conservation of energy and momentum the statistical properties of the system of discrete vortices have been examined. The model consists of an equal number of positive and negative vortices with equal absolute magnitudes $|\omega_i|=|\omega|$, in contact with a thermal bath of temperature T. The possibility arises for the generation of two supervortices of opposite signs, when T is large negative (i.e., negative temperatures). When the *most probable state* is attained for a stationary configuration the stream function ψ is shown to verify the sinh-Poisson equation (1). These statistical considerations (applicable also for $2D$ MHD or guiding center particles) remain the reference explanation for the appearance of this equation in this context [4–7].

In the equations of motion (3) the right hand side contains the *curl* of the Laplacian Green's function (4) (we take $L=1$)

$$-\varepsilon^{ij}\partial_j G(\mathbf{r},\mathbf{r}')=\varepsilon^{ij}\partial_j \frac{1}{2\pi}\ln r=\frac{1}{2\pi}\varepsilon^{ij}\frac{r^j}{r^2}$$
$$\nabla^2 \frac{1}{2\pi}\ln r=\delta^2(r). \tag{5}$$

The term in the right hand side of Eq. (3) can be considered as a vector potential $\mathbf{a}(\mathbf{r},t)$ whose "magnetic" field $\boldsymbol{\nabla}\times\mathbf{a}$ is a sum of Dirac δ functions at the locations of the vortices. If we take equal strength ω for all vortices this potential appears as the "statistical potential" and has a topological interpretation [8]. From Eq. (5) it can be rewritten as

$$\frac{1}{2\pi}\varepsilon^{ij}\frac{r^j}{r^2}=-\frac{1}{2\pi}\partial_i \arctan\frac{y}{x}=-\frac{1}{2\pi}\partial_i\theta, \tag{6}$$

where $\mathbf{r}=(x,y)=(r\cos\theta, r\sin\theta)$. The "magnetic" flux through a surface limited by a large circle is proportional to the number of vortices. The topological nature of this potential suggests that it can be naturally derived in a topological framework, i.e., from a Lagrangian density of the Chern-Simons type,

$$\mathcal{L}=\frac{1}{4}\varepsilon^{\mu\alpha\beta}A_\nu F_{\alpha\beta}, \tag{7}$$

where $F_{\alpha\beta}=\partial_\alpha A_\beta-\partial_\beta A_\alpha$. In the analysis of the two-dimensional Euler equation (as in the investigation of similar systems), the vorticity-type functions ω are expressed as the Lapacian operator applied on a scalar function and this naturally invokes the Green function of the Laplacian. Then this approach suggests that the intrinsically determined motion of the fluid can be projected onto two distinct parts of a new model: objects (vortices) and interaction between them (the potential obtained from the Green function). The fact that the interaction potential can be derived from a Chern-Simons topological action suggests that, in order to embed the original Euler fluid system into a larger field-theoretical context, we need (a) a "matter" part in the Lagrangian, which should provide the free dynamics of the vortices; (b) the Chern-Simons term, to describe the free dynamics of the field; (c) the interaction term of the (Chern-Simons) field and the matter. The main requirement to this field-theoretical extension of the original Euler fluid model is to reproduce the discrete vortices and their equation of motion.

Jackiw and Pi [9,10] have examined a model of N interacting particles (of charges e_s) moving in a plane described by the Lagrangian

$$L=\sum_{s=1}^{N}\frac{1}{2}m_s\mathbf{v}_s^2+\frac{1}{2}\int d^2r\,\varepsilon^{\alpha\beta\gamma}(\partial_\alpha A_\beta)A_\gamma-\int d^2r A_\mu j^\mu, \tag{8}$$

where $m_s\mathbf{v}_s=\mathbf{p}_s-e_s\mathbf{A}(\mathbf{r}_s|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)$, $\mathbf{A}(\mathbf{r}_s|\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)\equiv(a_s^i(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N))_{i=1,2}$, and $a_s^i(\mathbf{r}_1,\mathbf{r}_2,\ldots,\mathbf{r}_N)=1/2\pi\varepsilon^{ij}\Sigma_{q\neq s}^{N}e_q r_s^j-r_q^j/|\mathbf{r}_s-\mathbf{r}_q|^2$. The matter current is defined as $j^\mu\equiv(\rho,j)=\Sigma_{s=1}^{N}e_s v_s\delta(\mathbf{r}-\mathbf{r}_s)$, $v_s^\mu=(1,\mathbf{v}_s)$ with the metric $(1,-1,-1)$. By varying the action we obtain the following equations:

$$\frac{1}{2}\varepsilon^{\alpha\beta\gamma}F_{\alpha\beta}=\varepsilon^{\gamma\alpha\beta}\partial_\alpha A_\beta=j^\gamma,$$
$$B=-\rho, \tag{9}$$
$$E^i=\varepsilon^{ij}j^j. \tag{10}$$

Here $B=\varepsilon^{ij}\partial_j A_i$. In these equations the potential appears as being generated by the matter, i.e., by the "current" of particles. We note that Eq. (9) connects the matter density with the curl of the potential. This is important since in order to derive it in the context of the quantum version of their model, Jackiw and Pi have shown that one needs to include a nonlinear self-interaction of the wave function representing the matter field. The self-interaction of the scalar matter field is of the type φ^4. The canonical momentum expression becomes the covariant derivative in the field-theory version. Even at this point where we only have some hints, it is suggestive to look for possible identifications between the fluid variables and the field-theoretical variables of the model of Jackiw and Pi. The fluid variables are the scalar potential ψ, the velocity $v^i=\varepsilon^{ij}\partial_j\psi$, and the vorticity $\omega=\varepsilon^{ij}\partial_i v_j$. In order to check the possibility that the fluid model can be embedded into the field theory just described we identify

$$\omega\equiv\rho=\Psi^*\Psi, \tag{11}$$
$$v^i\equiv A^i. \tag{12}$$

Consider the relation between the vector potential $\mathbf{A}$ and the density ρ, obtained from Eq. (9),

$$\partial^i\partial_i A^j=-\varepsilon^{jk}\partial_k\rho$$

or, writing symbolically the Green function of the Laplace operator for argument $\mathbf{r}-\mathbf{r}'$ as the inverse of the operator at the left, we have

$$A^j(\mathbf{r},t)=\varepsilon^{jk}\partial_k\int d^2r'(-\partial^i\partial_i)^{-1}_{rr'}\rho(\mathbf{r}',t). \tag{13}$$

Since $\omega=-\nabla^2\psi$ (where $\nabla^2\equiv\partial^i\partial_i$) we have formally $\psi=-(\nabla^2)^{-1}\,\omega$ and use this formula to calculate the right side term of Eq. (13) taking into account the identification (11)

$$\varepsilon^{jk}\partial_k\int d^2r'(-\partial^i\partial_i)^{-1}_{rr'}\omega(\mathbf{r}')=\varepsilon^{jk}\partial_k\psi=v^j,$$

which confirms Eq. (12). It results that if we assume identification (11), Eq. (9) obtained from the Lagrangian is precisely the *definition* of the vorticity vector.

One important hint from the model of Jackiw and Pi is the idea to represent a classical quantity as the modulus of a fictitious complex scalar field, as in Eq. (11) and derive dynamical equations from a Lagrangian density expressed in terms of this field. The substitution of the dynamics expressed in terms of usual mechanical quantities by the richer dynamics of the amplitude and phase of the complex scalar field is an example of embedding of one theory into a larger framework and relies on the example of quantum mechanics.

In Ref. [11] the sinh-Poisson equation is derived in a field-theoretical model where instead of the topological coupling two complex scalar fields are considered. In that model the dynamics of the two scalar fields is independent except that the self-interaction depends on both. Combining this suggestion with that one exposed in the previous paragraph, we will take the density ρ [which we identify with the local value of the vorticity ω as in Eq. (11)] as being expressed by a field $\Psi \in SU(2)$ and depending on at least two complex scalar functions.

$$\rho \sim [\Psi^{\dagger},\Psi]. \tag{14}$$

Keeping the same structure of the Lagrangian density as in the previous example but extending to a non-Abelian SU(2) gauge field we have [12,10]

$$\mathcal{L} = -\varepsilon^{\mu\nu\rho}\mathrm{Tr}\left(\partial_{\mu}A_{\nu}A_{\rho}+\frac{2}{3}A_{\mu}A_{\nu}A_{\rho}\right)+i\mathrm{Tr}(\Psi^{\dagger}D_0\Psi) - \frac{1}{2}\mathrm{Tr}[(D_i\Psi)^{\dagger}D_i\Psi]+\frac{1}{4}\mathrm{Tr}([\Psi^{\dagger},\Psi])^2, \tag{15}$$

where the potential A takes values in the algebra of the group SU(2). This form incorporates and adapts all the suggestions from the previous models: (a) the first term is the general non-Abelian expression of the Chern-Simons term; (b) it uses the covariant derivatives ($\mu=0,1,2,\ i=1,2$) for the minimal coupling

$$D_{\mu}\Psi = \partial_{\mu}\Psi + [A_{\mu},\Psi].$$

(c) Finally, it includes a scalar self-interaction of φ^4 type. Analogous to the case treated by Jackiw and Pi for the U(1) gauge field (8), in Ref. [12] it is shown that the Hamiltonian density corresponding to the Lagrangian density (15) is

$$\mathcal{H} = \tfrac{1}{2}\mathrm{Tr}[(D_i\Psi)^{\dagger}(D_i\Psi)] - \tfrac{1}{4}\mathrm{Tr}([\Psi^{\dagger},\Psi]^2), \tag{16}$$

since the Chern-Simons term does not contribute to the energy density (being first order in the time derivatives). The equations of motion are

$$iD_0\Psi = -\frac{1}{2}\mathbf{D}^2\Psi - \frac{1}{2}([\Psi,\Psi^{\dagger}],\Psi), \tag{17}$$

$$F_{\mu\nu} = -\frac{i}{2}\varepsilon_{\mu\nu\rho}J^{\rho}. \tag{18}$$

Using the notation $D_{\pm}\equiv D_1 \pm iD_2$ the first term in Eq. (16) can be written as

$$\mathrm{Tr}[(D_i\Psi)^{\dagger}(D_i\Psi)] = \mathrm{Tr}[(D_-\Psi)^{\dagger}(D_-\Psi)] + \frac{1}{2}\mathrm{Tr}(\Psi^{\dagger}[[\Psi,\Psi^{\dagger}],\Psi]).$$

The last term comes from Eq. (18) and from the definition $J^0=[\Psi,\Psi^{\dagger}]$. Then the energy density is

$$\mathcal{H} = \tfrac{1}{2}\mathrm{Tr}[(D_-\Psi)^{\dagger}(D_-\Psi)] \geqslant 0 \tag{19}$$

and the Bogomol'nyi inequality is saturated at self-duality

$$D_-\Psi = 0, \tag{20}$$

$$\partial_+A_- - \partial_-A_+ + [A_+,A_-] = [\Psi,\Psi^{\dagger}]. \tag{21}$$

The first equation results from the minimum in Eq. (19) and the second is actually, Eq. (18), with the definitions

$$J^0 = [\Psi^{\dagger},\Psi],$$

$$J^i = -\frac{i}{2}\{[\Psi^{\dagger},D_i\Psi] - [(D_i\Psi)^{\dagger},\Psi]\}.$$

The *static* solutions of the *self-duality* equations (20) and (21) are derived in Ref. [12], using the algebraic ansatz:

$$A_i = \sum_{a=1}^{r} A_i^a H_a, \quad \Psi = \sum_{a=1}^{r} \psi^a E_a + \psi^M E_{-M},$$

where H_a are the Cartan subalgebra generators for the gauge Lie algebra, E_a are the simple-root step operators and E_{-M} is the step operator corresponding to the minus maximal root [13,14]. The rank of the algebra is noted r, and $r=1$ for SU (2). Then

$$[\Psi^{\dagger},\Psi] = \sum_{a=1}^{r} |\psi^a|^2 H_a + |\psi^M|^2 H_{-M}. \tag{22}$$

Equations (20) and (21) lead to the affine Toda equations

$$\nabla^2 \ln \rho_a + \sum_{b=1}^{r+1} \tilde{C}_{ab}\rho_b = 0 \tag{23}$$

for $a=1,r$, plus the index for M, i.e., $a=1,2$. $\tilde{C}_{ab}$ is the extended Cartan matrix

$$\tilde{C}_{ab} = \frac{2\alpha^{(a)}\cdot\alpha^{(b)}}{|\alpha^{(b)}|^2}, \quad a,b=1,2,$$

where $\alpha^{(a)}$ are the simple root vectors of the algebra su (2), and in addition the minus maximal root. Equations (23) can be written in detail for $\rho_1 \equiv |\psi^1|^2$, $\rho_2 \equiv |\psi^{-M}|^2$,

$$\begin{aligned} \Delta \ln \rho_1 + 2(\rho_1 - \rho_2) &= 0, \\ \Delta \ln \rho_2 + 2(-\rho_1 + \rho_2) &= 0, \end{aligned} \tag{24}$$

and this gives the relation

$$\Delta \ln(\rho_1\rho_2) = 0,$$

or

$$\rho_2 = \mathrm{const}\ \ \rho_1^{-1}, \tag{25}$$

since the exponential of a linear term is excluded by the conditions on a circle at infinity. Using Eq. (22) we have to identify

$$\omega = \rho_1 - \rho_2, \tag{26}$$

whose equation is obtained from Eqs. (24) and (25) with const equal to 1 (see below),

$$\Delta \ln \rho_1 + 2(\rho_1 - \rho_1^{-1}) = 0. \tag{27}$$

The substitution $\psi' \equiv \ln \rho_1$ transforms Eq. (27) in $-\Delta(\psi'/2) = 2\sinh(\psi')$ and Eq. (26) in $\omega = 2\sinh(\psi')$. This shows that $\psi'/2 \equiv \psi$ (the stream function) and we have $\Delta\psi + 2\sinh(2\psi) = 0$. Actually we can multiply ψ with an arbitary constant γ and/or put in front of sinh any other arbitrary constant β, since these can be absorbed in scalings of the space variables (x,y). We then have

$$\Delta\psi + \gamma \sinh(\beta\psi) = 0. \tag{28}$$

Choosing a constant different of unity in Eq. (25) imposes a linear substitution of the stream function: $\psi \rightarrow \psi' \equiv \gamma + \beta\psi$ and modifies the factor multiplying the sinh function in Eq. (28). The scaling we have mentioned allows us to adopt the simplest form of the sinh-Poisson equation, $\Delta\psi + \sinh\psi = 0$. The exact solution of the equation is $2\omega = [\theta(u+1/2,\tau)/\theta(u,\tau)]^2 - [\theta(u+1/2,\tau)/\theta(u,\tau)]^{-2}$. Here θ are Reimann *theta* functions, $u = k_x x + k_y y$. The parameters (k_x, k_y) and τ are determined in terms of the main spectrum of the Lax eigenproblem and the periods of a basis of differential one forms on the elements of the basis of the first cohomology group of a Riemann surface [15]. At self-duality [i.e., states described by Eq. (28)] the 2D fluid invariants $\int \omega^n dx dy$ are all connected: the nth order can be derived by recurrence from lower orders, since $\sinh^n \psi$ can be expressed through $\sinh(m\psi)$, $m \leqslant n$, followed by a scaling as mentioned above.

We have constructed the Lagrangian density (15) with a standard structure of matter-gauge field interaction, and incorporating suggestions from models relevant to our purpose. The aim was to reproduce the model of fluid vortex filaments supposed to be a faithful representation of the original Euler fluid dynamics. The fundamental step in the derivation of the sinh-Poisson equation is the assumption that energy density (19) is minimum (which is equivalent to minimizing the *action* for these stationary solution), i.e., in the condition of self-duality. The self-duality can only be revealed in this framework and this sheds a new light on the asymptotic states consisting of regular vortices. Conversely, we can expect that the reductions from self-dual Yang-Mills (via twistors) to the exactly integrable hierarchies and Painleve transcendents have a correspondence in the states of maximum probability of a finite statistical model with various conservation constraints.

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